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Chebyshev's maximum principle in several variables ☆

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Abstract

In this short note, we discuss the Chebyshev's maximum principle in several variables. We show some analogous maximum formulae for the common zeros in some special cases. It can be regarded as the extension of the univariate case.

Keywords: Multivariate orthonormal polynomials; Reproducing kernel

Let ω be a nonnegative weight function on $(-\infty, \infty)$ for which $x^n\omega(x) \in L^1(-\infty, \infty)$ $(n = 0, 1, \dots)$. We construct the sequence of orthonormal polynomials $p_n(\omega; x) = \gamma_n(\omega)x^n + \cdots + (\gamma_n(\omega) > 0)$, satisfying

$$\int_{-\infty}^{\infty} p_m(\omega; x)p_n(\omega; x)\omega(x)\,dx = \delta_{mn}.$$

It is well known that all zeros x_{kn} of $p_n(\omega)$ are real and simple. Let us denote by $X_n(\omega)$ the greatest zero of $p_n(\omega)$. The sequence $\{X_n(\omega)\}$ is increasing, and, by virtue of a result of Chebyshev, (see [2–5]),

$$X_n(\omega) = \max_{\pi_{n-1} \in \Pi^1_{n-1}, \pi_{n-1} \neq 0} \frac{\int_{-\infty}^{\infty} x\pi^2_{n-1}(x)\omega(x)\,dx}{\int_{-\infty}^{\infty} \pi^2_{n-1}(x)\omega(x)\,dx} \quad (n = 1, 2, \dots), \tag{1}$$

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where Π^1_{n-1} denotes the subspace of polynomials whose degree is not larger than $n-1$. It is the so-called Chebyshev's maximum principle.

The essence of relation (1) lies in the fact that the Gauss cubature formula for one variable always exists. But it is not always true for several variables. In the following, we will discuss the analogous relationship as (1) in the multivariate case under some conditions. For convenience, some basic notations and results for several variables are presented here.

Let Π^d be the space of all polynomials in d variables, Π^d_n the subspace of polynomials of total degree not larger than n in d variables. Let L be a positive definite linear functional acting on Π^d, that is,

$$L(p^2)>0, \quad \forall p\in\Pi^d, \quad p\neq 0.$$

Let $\{P^n_\alpha\}_{|\alpha|=n}$ be a sequence of orthonormal polynomials with respect to L, $K_n(x,y)=\sum_{k=0}^n \sum_{|\alpha|=k} P^k_\alpha(x)P^k_\alpha(y)$ the nth reproducing kernel.

Definition 1 (Dunkl and Yuan Xu [1]). A cubature formula of degree $2n-1$ with $dim\,\Pi^d_{n-1}$ nodes is called a Gaussian cubature.

Definition 2. A positive definite linear functional L acting on Π^d is called Gaussian, if $\{P^n_\alpha\}_{|\alpha|=n}$ has $N=dim\,\Pi^d_{n-1}$ common zeros.

Lemma 1 (Dunkl and Yuan Xu [1]). *Let L be a positive definite linear functional. Then L admits a Gaussian cubature formula of degree $2n-1$ if and only if $\{P^n_\alpha\}_{|\alpha|=n}$ has $N=dim\,\Pi^d_{n-1}$ common zeros.*

In what follows, we shall always assume L is Gaussian, and $H_n=\{X^n_i=(x^n_{i1},\dots,x^n_{id})\}_{i=1}^N$ are the mutually different common zeros of $\{P^n_\alpha\}_{|\alpha|=n}$.

Theorem 1. *Let $l(x_1,\dots,x_d)$ be an arbitrary linear function with respect to $\{x_i\}_{i=1}^d$. Then*

$$\max\{l(X^n_i), \quad X^n_i\in H_n\} = \max_{\pi_{n-1}\in\Pi^d_{n-1},\pi_{n-1}\neq 0} \frac{L(l(x)\pi^2_{n-1}(x))}{L(\pi^2_{n-1}(x))}.$$

Proof. Since L is Gaussian, we have

$$L(P(x))=\sum_{i=1}^N \rho^n_i P(X^n_i), \quad \rho^n_i>0 \ (i=1,\dots,N),$$

for all $P(x)\in\Pi^d_{2n-1}$. Therefore, for arbitrary $\pi_{n-1}\in\Pi^d_{n-1}$,

$$\begin{aligned} L(l(x)\pi^2_{n-1}(x)) &= \sum_{i=1}^{N} \rho_i^n l(X_i^n)\pi^2_{n-1}(X_i^n) \\ &\leqslant \max\{l(X_i^n), X_i^n\in H_n\}L(\pi^2_{n-1}(x)), \end{aligned}$$

that is,

$$\max\{l(X_i^n), X_i^n\in H_n\} \geqslant \max_{\pi_{n-1}\in\Pi^d_{n-1},\pi_{n-1}\neq 0} \frac{L(l(x)\pi^2_{n-1}(x))}{L(\pi^2_{n-1}(x))}. \tag{2}$$

On the other hand, let

$$l_k(x) = \frac{K_{n-1}(x, X_k^n)}{K_{n-1}(X_k^n, X_k^n)}, \quad k = 1, \dots, N,$$

which is meaningful by $K_{n-1}(x, y)$ being always positive.

Then, we have

$$deg(l_k(x)) = n-1, \quad l_k(X_j^n) = \delta_{k,j}, \quad 1\leqslant k,j\leqslant N.$$

Choosing k to be the index which satisfies $l(X_k^n) = \max\{l(X_i^n), X_i^n\in H_n\}$, it is easy to see that

$$\max\{l(X_i^n), X_i^n\in H_n\} = l(X_k^n) = \frac{L(l(x)l_k^2(x))}{L(l_k^2(x))}. \tag{3}$$

Combining (2) and (3), we can obtain the result immediately. $\square$

Remark 1. By Theorem 1, it is obvious that the sequence $\{\max\{l(X_i^n), X_i^n\in H_n\}\}_{n=1}^{\infty}$ is nondecreasing.

As some special cases of Theorem 1, we have the following.

Corollary 1.

$$\max\{x_{ij}^n, X_i^n\in H\} = \max_{\pi_{n-1}\in\Pi^d_{n-1},\pi_{n-1}\neq 0} \frac{L(x_j\pi^2_{n-1}(x))}{L(\pi^2_{n-1}(x))}, \quad j = 1, \dots, d.$$

Corollary 2.

$$\max\left\{\sum_{j=1}^{d} x_{ij}^n, X_i^n\in H_n\right\} = \max_{\pi_{n-1}\in\Pi^d_{n-1},\pi_{n-1}\neq 0} \frac{L(\sum_{j=1}^d x_j\pi^2_{n-1}(x))}{L(\pi^2_{n-1}(x))}.$$

Corollary 3.

$$\max\{|x_{ij}^n|, X_i^n \in H_n\} = \max\left\{ \max_{\pi_{n-1}\in\Pi_{n-1}^d, \pi_{n-1}\neq 0} \frac{L(x_j\pi_{n-1}^2(x))}{L(\pi_{n-1}^2(x))}, \max_{\pi_{n-1}\in\Pi_{n-1}^d, \pi_{n-1}\neq 0} \frac{L(-x_j\pi_{n-1}^2(x))}{L(\pi_{n-1}^2(x))} \right\}, \quad j = 1, 2, \dots, d.$$

Corollary 4.

$$\max\left\{ \sum_{j=1}^d |x_{ij}^n|, X_i^n \in H_n \right\} = \max_{\varepsilon_j=\pm 1, j=1,\dots,d} \left\{ \max_{\pi_{n-1}\in\Pi_{n-1}^d, \pi_{n-1}\neq 0} \frac{L(\sum_{j=1}^d \varepsilon_j x_j \pi_{n-1}^2(x))}{L(\pi_{n-1}^2(x))} \right\}.$$

Remark 2. It is worth noting that the Gaussian cubature formulae in several variables rarely exist although they indeed exist in some special cases.

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